

Parallel Dynamics of Fully Connected Q -Ising Neural Networks

D. Bollé,^{1,2} G. Jongen,^{1,2} and G. M. Shim³

Received March 18, 1997; final December 8, 1997

Using a probabilistic approach, the parallel dynamics of fully connected Q -Ising neural networks is studied for arbitrary Q . A novel recursive scheme is set up to determine the time evolution of the order parameters through the evolution of the distribution of the local field, taking into account *all* feedback correlations. In contrast to extremely diluted and layered network architectures, the local field is no longer normally distributed but contains a discrete part. As an illustrative example, an explicit analysis is carried out for the first four time steps. For the case of the $Q=2$ and $Q=3$ model the results are compared with extensive numerical simulations and excellent agreement is found. Finally, equilibrium fixed-point equations are derived and compared with the thermodynamic approach based upon the replica-symmetric mean-field approximation.

KEY WORDS: Fully connected networks; Q -Ising neurons; parallel dynamics; probabilistic approach.

1. INTRODUCTION

The parallel dynamics of extremely diluted asymmetric and layered feed-forward $Q \geq 2$ -Ising neural networks have been solved exactly (cf. refs. 2–4 and the references cited therein). This has been possible because in these types of networks one knows that there are no feedback loops as time progresses. In particular, this allows one to derive recursion relations for the relevant order parameters of these systems: the main overlap for the

¹ Instituut voor Theoretische Fysica, K.U. Leuven, B-3001 Leuven, Belgium; e-mail: {desire.bolle, greetje.jongen}@fys.kuleuven.ac.be.

² Also at Interdisciplinair Centrum voor Neurale Netwerken, K.U. Leuven, Leuven, Belgium.

³ Department of Physics, Chungnam National University, Yuseong, Taejon 305-764, Korea; e-mail: gmshim@nsphys.chungnam.ac.kr.

condensed pattern, the mean of the neuron activities and the variance of the residual overlap responsible for the intrinsic noise in the dynamics of the main overlap (sometimes called the width-parameter).

These results are in strong contrast with those for the parallel dynamics of networks with symmetric connections. For these systems it turns out that even in the diluted $Q=2$ case, feedback correlations become essential from the second time step onwards, which already complicates the dynamics in a nontrivial way.⁽⁵⁻⁶⁾ For fully connected $Q=2$ systems, an increasing complexity of such long-term temporal correlations makes the dynamics, in general, extremely complicated. Therefore, either approximate treatments of the feedback influence on the network evolution or only the first two time steps of the main overlap evolution have been analyzed so far in the literature. Nevertheless, this has led to some important insights into the dynamics of the Little-Hopfield model (cf. refs. 7-15 and references therein).

In this paper we present a complete solution of this temporal feedback correlations problem in the zero-temperature parallel dynamics of fully connected Q -Ising neural networks not only for $Q=2$ but for general Q .

Using a probabilistic approach⁽¹³⁾ we are able to generalize our analysis for extremely diluted^(2, 16) and layered Q -Ising networks⁽⁴⁾ to the non-trivial case of fully connected systems at zero temperature. In particular, we succeed in developing a novel recursive scheme in order to calculate the relevant order parameters of the system, i.e., the main overlap, the activity and the variance of the residual overlap, for any time step and taking into account *all* feedback correlations. The main difference with our previous treatments is that we have to start from a study of the evolution of the distribution of the local field instead of working directly with the order parameters. In contrast with extremely diluted asymmetric and layered network architectures, we find that the local field is no longer normally distributed but contains a discrete part.

As an illustration we write out these expressions in detail for the first four time steps of the dynamics. Extensive numerical simulations for the $Q=2$ and $Q=3$ model are compared with the theoretical results. They confirm the exactness of this recursive scheme and do give a clear picture of the time evolution in the retrieval regime of the network.

Finally, generalizing a $Q=2$ result from the literature⁽¹⁷⁻¹⁸⁾ we find that there exists a Lyapunov function leading to the occurrence of fixed-points and two-cycles. Since two-cycles in the $Q=2$ Little-Hopfield model seem to appear far from the retrieval region⁽¹⁹⁾ and/or seem to involve only a tiny fraction of all spins,⁽²⁰⁾ we have only looked at the fixed-point dynamics. We show that the fixed-point equations for the order parameters derived via thermodynamical methods⁽²¹⁾ can be found from the dynamical

scheme presented here by requiring the local field to become time-independent implying that most of the discrete noise part is neglected.

The rest of the paper is organized as follows. In Section 2 we introduce the model, its dynamics and the Hamming distance as a macroscopic measure for the retrieval quality. In Section 3 we use the probabilistic approach in order to derive the new recursive scheme for the evolution of the distribution of the local field, leading to recursion relations for the order parameters of the model. Using this general scheme, we explicitly calculate in Section 4 the order parameters for the first four time steps of the dynamics. In Section 5 we show the existence of a Lyapunov function at zero temperature and we discuss the evolution of the system to fixed-point attractors. A detailed discussion of the theoretical results obtained in Section 4 and a comparison with extensive numerical simulations for $Q=2, 3$ are presented in Section 6. Some concluding remarks are given in Section 7.

2. THE MODEL

Consider a neural network A consisting of N neurons which can take values σ_i from a discrete set $\mathcal{S} = \{-1 = s_1 < s_2 < \dots < s_Q = +1\}$. Given the configuration $\boldsymbol{\sigma}_A(t) \equiv \{\sigma_j(t)\}$, $j \in A = \{1, \dots, N\}$, the local field in neuron i equals

$$h_i(\boldsymbol{\sigma}_{A \setminus \{i\}}(t)) = \sum_{j \in A \setminus \{i\}} J_{ij} \sigma_j(t) \quad (1)$$

with J_{ij} the synaptic couplings between neurons i and j . In the sequel we write the shorthand notation $h_{A,i}(t) \equiv h_i(\boldsymbol{\sigma}_{A \setminus \{i\}}(t))$. The configuration $\boldsymbol{\sigma}_A(t=0)$ is chosen as input. At zero temperature all neurons are updated in parallel according to the rule

$$\sigma_i(t) \rightarrow \sigma_i(t+1) = s_k : \min_{s \in \mathcal{S}} \varepsilon_i[s | \boldsymbol{\sigma}_{A \setminus \{i\}}(t)] = \varepsilon_i[s_k | \boldsymbol{\sigma}_{A \setminus \{i\}}(t)] \quad (2)$$

Here the energy potential $\varepsilon_i[s | \boldsymbol{\sigma}_{A \setminus \{i\}}]$ is defined by

$$\varepsilon_i[s | \boldsymbol{\sigma}_{A \setminus \{i\}}] = -\frac{1}{2} [h_i(\boldsymbol{\sigma}_{A \setminus \{i\}}) s - b s^2] \quad (3)$$

where $b > 0$ is the gain parameter of the system. The updating rule (2) is equivalent to using a gain function $\mathbf{g}_b(\cdot)$,

$$\begin{aligned} \sigma_i(t+1) &= \mathbf{g}_b(h_{A,i}(t)) \\ \mathbf{g}_b(x) &\equiv \sum_{k=1}^Q s_k [\theta[b(s_{k+1} + s_k) - x] - \theta[b(s_k + s_{k-1}) - x]] \end{aligned} \quad (4)$$

with $s_0 \equiv -\infty$ and $s_{Q+1} \equiv +\infty$. For finite Q , this gain function $g_b(\cdot)$ is a step function. The gain parameter b controls the average slope of $g_b(\cdot)$.

In this network we want to store $p = \alpha N$ patterns. These patterns are a collection of independent and identically distributed random variables (i.i.d.r.v.), $\{\xi_i^\mu \in \mathcal{S}\}$, $\mu \in \mathcal{P} = \{1, \dots, p\}$ and $i \in \mathcal{A}$ with zero mean and variance $A = \text{Var}[\xi_i^\mu]$. The synaptic couplings between the neurons are chosen according to the Hebb learning rule

$$J_{ij} = \frac{1}{NA} \sum_{\mu \in \mathcal{P}} \xi_i^\mu \xi_j^\mu \quad \text{for } i \neq j, \quad J_{ii} = 0 \quad (5)$$

To measure the retrieval quality of the system one can use the Hamming distance between a stored pattern and the microscopic state of the network

$$d(\xi^\mu, \sigma_{\mathcal{A}}(t)) \equiv \frac{1}{N} \sum_{i \in \mathcal{A}} [\xi_i^\mu - \sigma_i(t)]^2 \quad (6)$$

This naturally introduces the main overlap

$$m_{\mathcal{A}}^\mu(t) = \frac{1}{NA} \sum_{i \in \mathcal{A}} \xi_i^\mu \sigma_i(t) \quad \mu \in \mathcal{P} \quad (7)$$

and the arithmetic mean of the neuron activities

$$a_{\mathcal{A}}(t) = \frac{1}{N} \sum_{i \in \mathcal{A}} [\sigma_i(t)]^2 \quad (8)$$

We remark that for $Q=2$ the variance of the patterns $A=1$, and the neuron activity $a_{\mathcal{A}}(t)=1$.

3. GENERAL DYNAMICAL SCHEMA

It is known that contrary to the asymmetrically diluted and layered neural networks, the parallel dynamics of fully connected systems, even at zero temperature, is difficult to solve because of the strong feedback correlations.⁽²²⁾

On the basis of the probabilistic approach used before (see, e.g., refs. 4 and 13) we develop in this section a recursive dynamical scheme in order to calculate the distribution of the local field at a general time step, for

$Q \geq 2$ -Ising neural networks. This results in recursion relations determining the full time evolution of the order parameters of these systems.

Suppose that the initial configuration of the network $\{\sigma_i(0)\}$, $i \in A$, is a collection of i.i.d.r.v. with mean $E[\sigma_i(0)] = 0$, variance $\text{Var}[\sigma_i(0)] = a_0$, and correlated with only one stored pattern, say the first one $\{\xi_i^1\}$:

$$E[\xi_i^\mu \sigma_i(0)] = \delta_{\mu,1} m_0^1 A \quad m_0^1 > 0 \quad (9)$$

This implies that by the law of large numbers (LLN) one gets for the main overlap and the activity at $t=0$

$$m^1(0) \equiv \lim_{N \rightarrow \infty} m_{A^1}^1(0) \stackrel{Pr}{\cong} \frac{1}{A} E[\xi_i^1 \sigma_i(0)] = m_0^1 \quad (10)$$

$$a(0) \equiv \lim_{N \rightarrow \infty} a_A(0) \stackrel{Pr}{\cong} E[\sigma_i^2(0)] = a_0 \quad (11)$$

where the convergence is in probability.⁽²³⁾ Using standard signal-to-noise techniques (see, e.g., ref. 16), we find the local field at $t=0$

$$\begin{aligned} h_i(0) &= \lim_{N \rightarrow \infty} \left[\xi_i^1 m_{A \setminus \{i\}}^1(0) + \frac{1}{NA} \sum_{\mu \in \mathcal{P} \setminus \{1\}} \sum_{j \in A \setminus \{i\}} \xi_i^\mu \xi_j^\mu \sigma_j(0) \right] \\ &\stackrel{\mathcal{D}}{\cong} \xi_i^1 m^1(0) + \mathcal{N}(0, \alpha a_0), \end{aligned} \quad (12)$$

where the convergence is in distribution (see, e.g., ref. 23). The quantity $\mathcal{N}(0, d)$ represents a Gaussian random variable with mean 0 and variance d .

For a general time step we find from Eq. (4) and the LLN in the limit $N \rightarrow \infty$ for the main overlap (7) and the activity (8)

$$m^1(t+1) \stackrel{Pr}{\cong} \frac{1}{A} \langle\langle \xi_i^1 \mathbf{g}_b(h_i(t)) \rangle\rangle \quad (13)$$

$$a(t+1) \stackrel{Pr}{\cong} \langle\langle \mathbf{g}_b^2(h_i(t)) \rangle\rangle \quad (14)$$

with $h_i(t) \equiv \lim_{N \rightarrow \infty} h_{A,i}(t)$. In the above $\langle\langle \cdot \rangle\rangle$ denotes the average both over the distribution of the embedded patterns $\{\xi_i^\mu\}$ and the initial configurations $\{\sigma_i(0)\}$. The average over the initial configurations is hidden in an average over the local field through the updating rule (4). From the study of layered networks⁽³⁻⁴⁾ we know already that due to the correlations there will be a third important parameter in the description of the time evolution of the system: the influence of the non-condensed patterns which is expressed by the variance of the residual overlaps

$$D(t) \equiv \text{Var}[r^\mu(t)] \quad (15)$$

$$\begin{aligned} r^\mu(t) &\equiv \lim_{N \rightarrow \infty} r_{\mathcal{A}}^\mu(t) \\ &= \lim_{N \rightarrow \infty} \frac{1}{A\sqrt{N}} \sum_{i \in \mathcal{A}} \xi_i^\mu \sigma_i(t) \quad \mu \in \mathcal{P} \setminus \{1\} \end{aligned} \quad (16)$$

Clearly, $D(0) = a_0/A$.

It is the purpose of this section to calculate the distribution of the local field as a function of time. This calculation can be done for arbitrary Q without making the notation much heavier. The simplifications for $Q=2$ will be mentioned at the end.

We start by rewriting the local field (1) at time t in the following way

$$h_{\mathcal{A},i}(t) = \xi_i^1 m_{\mathcal{A}}^1(t) + \frac{1}{NA} \sum_{j \in \mathcal{A}} \sum_{\mu \in \mathcal{P} \setminus \{1\}} \xi_i^\mu \xi_j^\mu \sigma_j(t) - \alpha \sigma_i(t) \quad (17)$$

$$= \xi_i^1 m_{\mathcal{A}}^1(t) - \alpha \sigma_i(t) + \frac{1}{\sqrt{N}} \sum_{\mu \in \mathcal{P} \setminus \{1\}} \xi_i^\mu r_{\mathcal{A}}^\mu(t) \quad (18)$$

From a technical point of view the explicit addition and subtraction of the $\alpha \sigma_i(t)$ term in (17) is convenient in order to treat all indices in the sum over j on an equal footing. This turns out to be important to take into account all possible feedback loops.

In order to determine the structure of the local field for fully connected networks, we first concentrate on the residual overlap $r_{\mathcal{A}}^\mu(t)$, $\mu \in \mathcal{P} \setminus \{1\}$. At this point we would like to remark that the set of αN variables $\{\xi_i^\mu r_{\mathcal{A}}^\mu(t)\}_\mu$ appearing in the last term of (18) are not independent because the ξ_i^μ are weakly dependent on the $r_{\mathcal{A}}^\nu(t)$, $\nu \neq \mu$. Indeed, after applying the dynamics the $\sigma_i(t)$ and the ξ_i^μ become dependent, leading to a weak dependence of $r_{\mathcal{A} \setminus \{i\}}^\mu(t)$ and ξ_i^μ . This microscopic dependence gives rise to a macroscopic contribution after summing and taking the limit $N \rightarrow \infty$.

On the contrary, in the case of layered or diluted networks all terms of this set of variables are independent such that their sum is a normal distribution. Moreover, $r_{\mathcal{A} \setminus \{i\}}^\mu(t)$ and ξ_i^μ are also independent implying that the mean and the variance of this distribution are known directly. This has been used to put forward approximations for the fully connected $Q=2$ model, precisely by neglecting (some of) these correlations between the ξ_i^μ and the $r_{\mathcal{A} \setminus \{i\}}^\mu(t)$.^(6,9) Naturally, the results obtained could not be verified by numerical simulations and further improvements are, up to now, not satisfactory.^(5,15) So a complete treatment of these correlations is necessary.

In order to study the contribution from the dependence between the ξ_i^μ and the $r_{A \setminus \{i\}}^\mu(t)$ we rewrite the residual overlap (16) as

$$r_A^\mu(t+1) = \frac{1}{\sqrt{N} A} \sum_{i \in A} \xi_i^\mu \mathbf{g}_b \left(\hat{h}_{A,i}^\mu(t) + \frac{1}{\sqrt{N}} \xi_i^\mu r_A^\mu(t) \right) \quad (19)$$

with

$$\hat{h}_{A,i}^\mu(t) = h_{A,i}(t) - \frac{1}{\sqrt{N}} \xi_i^\mu r_A^\mu(t) \quad (20)$$

By subtracting the term $\xi_i^\mu r_A^\mu(t)/\sqrt{N}$ the modified local field $\hat{h}_{A,i}^\mu(t)$ becomes only weakly dependent on ξ_i^μ , whereas $h_{A,i}(t)$ depends strongly on ξ_i^μ . We next want to expand $\mathbf{g}_b(\cdot)$ around $\hat{h}_{A,i}^\mu(t)$. Therefore we have to find out when the term $\xi_i^\mu r_A^\mu(t)/\sqrt{N}$ in $\mathbf{g}_b(\cdot)$ becomes relevant. Since the gain function $\mathbf{g}_b(\cdot)$ is a step function that changes its value by $s_{k+1} - s_k$ at $b(s_k + s_{k+1})$, $k = 1, \dots, Q-1$ we find that it is relevant if for some k

$$|\hat{h}_{A,i}^\mu(t) - b(s_k + s_{k+1})| < \frac{|\xi_i^\mu r_A^\mu(t)|}{\sqrt{N}} \quad k = 1, \dots, Q-1 \quad (21)$$

Denoting by I_k the set of indices satisfying condition (21), we split the residual overlap into two sums:

$$\begin{aligned} r_A^\mu(t+1) &= \frac{1}{\sqrt{N} A} \sum_{i \notin \cup_k I_k} \xi_i^\mu \mathbf{g}_b(\hat{h}_{A,i}^\mu(t)) \\ &\quad + \frac{1}{\sqrt{N} A} \sum_{k=1}^{Q-1} \sum_{i \in I_k} \xi_i^\mu \left\{ \mathbf{g}_b(\hat{h}_{A,i}^\mu(t)) \right. \\ &\quad \left. + \frac{1}{2} \left[\text{sign}(b(s_k + s_{k+1}) - \hat{h}_{A,i}^\mu(t)) \right. \right. \\ &\quad \left. \left. + \text{sign} \left(\frac{1}{\sqrt{N}} \xi_i^\mu r_A^\mu(t) \right) \right] (s_{k+1} - s_k) \right\} \quad (22) \end{aligned}$$

In the argument of $\mathbf{g}_b(\cdot)$ in the first term of (22) the term $\xi_i^\mu r_A^\mu(t)/\sqrt{N}$ is left out since it can not change the value of $\mathbf{g}_b(\cdot)$ by definition of the sets I_k . Combining the first two terms, Eq. (22) can be rewritten as

$$\begin{aligned}
r_A^\mu(t+1) &= \frac{1}{\sqrt{N}} \sum_{i \in A} \xi_i^\mu \mathbf{g}_b(\hat{h}_{A,i}^\mu(t)) \\
&\quad + \frac{1}{\sqrt{N}} \sum_{k=1}^{Q-1} \sum_{i \in I_k} \xi_i^\mu \frac{1}{2} \left[\text{sign}(b(s_k + s_{k+1}) - \hat{h}_{A,i}^\mu(t)) \right. \\
&\quad \left. + \text{sign}\left(\frac{1}{\sqrt{N}} \xi_i^\mu r_A^\mu(t)\right) \right] (s_{k+1} - s_k) \tag{23}
\end{aligned}$$

This completes the rewriting of \mathbf{g}_b .

We then consider the limit $N \rightarrow \infty$. In this limit the cardinal number of the set I_k becomes deterministic

$$\lim_{N \rightarrow \infty} \frac{|I_k|}{\sqrt{N}} = 2 |\xi_i^\mu r^\mu(t)| f_{\hat{h}_i^\mu(t)}(b(s_{k+1} + s_k)) \tag{24}$$

with $f_{\hat{h}_i^\mu(t)}$ the probability density of the modified local field $\hat{h}_i^\mu(t)$ at time t . We remark that in the thermodynamic limit the density distribution of the modified local field $\hat{h}_i^\mu(t)$ at time t equals the density distribution of the local field $h_i(t)$ itself. Furthermore, we apply the CLT on the first term of (23) and the LLN on the second term with the random variable $r_A^\mu(t)$ fixed. This yields the following result

$$r^\mu(t+1) = \tilde{r}^\mu(t) + \chi(t) r^\mu(t) \tag{25}$$

where, recalling Eqs. (14) and (20)

$$\tilde{r}^\mu(t) \equiv \lim_{N \rightarrow \infty} \frac{1}{A \sqrt{N}} \sum_{i \in A} \xi_i^\mu \mathbf{g}_b(\hat{h}_{A,i}^\mu(t)) \stackrel{\mathcal{D}}{=} \mathcal{N}_\mu(0, a(t+1)/A) \tag{26}$$

because of the weak dependence of $\hat{h}_i^\mu(t)$ and ξ_i^μ , and

$$\chi(t) = \sum_{k=1}^{Q-1} f_{\hat{h}_i^\mu(t)}(b(s_{k+1} + s_k))(s_{k+1} - s_k) \tag{27}$$

From the relation (25) one finds a recursion relation for the variance of the residual overlap

$$D(t+1) = \frac{a(t+1)}{A} + \chi^2(t) D(t) + 2\chi(t) \text{Cov}[\tilde{r}^\mu(t), r^\mu(t)] \tag{28}$$

This finishes the treatment of the residual overlap. We first remark that for $Q=2$ the following simplifications are possible: $b(s_{k+1} + s_k) = 0$, $(s_{k+1} - s_k) = 2$, $g_b(\cdot) = \text{sign}(\cdot)$ and $a(t+1)/A = 1$. Furthermore, for all Q the last term on the r.h.s. of (28) is entirely coming from the correlations caused by the fully connected structure of the network. It is absent for layered (compare Eq. (30c) of ref. 4) and hence, of course, also for extremely diluted asymmetric architectures. In the latter case also the second term on the r.h.s. of (28) disappears.

Next, starting from the local field at time $t+1$ in the form of Eq. (18) and using expressions (25), (26) and (27) we obtain in the limit $N \rightarrow \infty$, after some straightforward manipulations

$$h_i(t+1) = \xi_i^1 m^1(t+1) + \chi(t)[h_i(t) - \xi_i^1 m^1(t) + \alpha \sigma_i(t)] + \mathcal{N}(0, \alpha a(t+1)) \quad (29)$$

From this it is clear that the local field at time t consists out of a discrete part and a normally distributed part, viz.

$$h_i(t) = M_i(t) + \mathcal{N}(0, V(t)) \quad (30)$$

where $M_i(t)$ satisfies the recursion relation

$$M_i(t+1) = \chi(t)[M_i(t) - \xi_i^1 m^1(t) + \alpha \sigma_i(t)] + \xi_i^1 m^1(t+1) \quad (31)$$

and

$$V(t+1) = \alpha A D(t+1) \quad (32)$$

with $D(t+1)$ given by the recursion relation (28).

We still have to determine $f_{h_i(t)}$ in Eq. (27). We know that the quantity $M_i(t)$ consists out of the signal term and a discrete noise term, viz.

$$M_i(t) = \xi_i^1 m^1(t) + \sum_{t'=0}^{t-1} \alpha \left[\prod_{s=t'}^{t-1} \chi(s) \right] \sigma_i(t') \quad (33)$$

The evolution equation tells us that $\sigma_i(t')$ can be replaced by $g_b(h_i(t'-1))$ such that the second term of $M_i(t)$ is the sum of stepfunctions of correlated variables. These are also correlated through the dynamics with the normally distributed part of $h_i(t)$. Therefore the local field can be considered as a transformation of a set of correlated normally distributed variables x_s , $s=0, \dots, t-2, t$, which we choose to normalize. Defining the correlation

matrix $\rho(s, s') \equiv E[x_s x_{s'}]$ we arrive at the following expression for the probability density of the local field at time t

$$f_{h_i(t)}(y) = \int \prod_{s=0}^{t-2} dx_s dx_t \delta(y - M_i(t) - \sqrt{V(t)} x_t) \times \frac{1}{\sqrt{\det(2\pi C)}} \exp\left(-\frac{1}{2} \mathbf{x} C^{-1} \mathbf{x}^T\right) \quad (34)$$

with $\mathbf{x} = (x_0, \dots, x_{t-2}, x_t)$.

Together with the Eqs. (13)–(14) for $m^1(t+1)$ and $a(t+1)$ the equations (25)–(28), (31) and (34) form a recursive scheme in order to obtain the order parameters of the system. The practical difficulty which remains is the explicit calculation of the correlations in the network at different time steps as present in Eq. (28).

4. EVOLUTION EQUATIONS UP TO THE FOURTH TIME STEP

Following the general recursive scheme established in Section 3 evolution equations are derived for the order parameters of a fully connected Q -Ising network for the first four time steps, taking into account all correlations. This generalizes and extends the $Q=2$ results in the literature mentioned in the Introduction.

4.1. First Step Dynamics

Starting from Eqs. (12), (13) and (14) one has immediately

$$m^1(1) = \frac{1}{A} \left\langle\left\langle \xi^1 \int \mathcal{D}z \mathbf{g}_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} z) \right\rangle\right\rangle \quad (35)$$

$$a(1) = \left\langle\left\langle \int \mathcal{D}z \mathbf{g}_b^2(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} z) \right\rangle\right\rangle \quad (36)$$

where $\langle\langle \dots \rangle\rangle$ now stands for the average taken with respect to the distribution of the first pattern and the initial configuration and $\mathcal{D}z$ denotes a Gaussian measure $\mathcal{D}z = dz \exp(-\frac{1}{2}z^2)/\sqrt{2\pi}$. We recall that $D(0) = a_0/A$. Next, from the initial conditions (9)–(12) and the definition of the modified local field (20) one also knows that $\{\xi_i^\mu, \mathbf{g}_b(\hat{h}_i^\mu(0))\}_i$ and $\{\xi_i^\mu, \sigma_i(0)\}_i$ become a set of uncorrelated parameters for $\mu \in \mathcal{P} \setminus \{1\}$. Here $\hat{h}_i^\mu(t) \equiv \lim_{N \rightarrow \infty} \hat{h}_{A,i}^\mu(t)$. Therefore

$$\text{Cov}[\tilde{r}^\mu(0), r^\mu(0)] = E[\sigma_i(0) \mathbf{g}_b(\hat{h}_i^\mu(0))] \quad (37)$$

Using the recursion relation (28) this leads to

$$D(1) = \frac{a(1)}{A} + \chi^2(0) D(0) + 2 \frac{\chi(0)}{A} \left\langle\left\langle \sigma(0) \int \mathcal{D}z z g_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} z) \right\rangle\right\rangle \quad (38)$$

with

$$\chi(0) = \left\langle\left\langle \frac{1}{\sqrt{\alpha AD(0)}} \int \mathcal{D}z z g_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} z) \right\rangle\right\rangle \quad (39)$$

These results generalize the corresponding $Q=2$ results (see, e.g., refs. 7 and 13).

4.2. Second Step Dynamics

First we need the distribution of the local field at time $t=1$. This follows immediately from Eqs. (31) and (32)

$$h_i(1) = \xi_i^1 m^1(1) + \chi(0) [h_i(0) - \xi_i^1 m^1(0) + \alpha \sigma_i(0)] + V(0, \alpha a(1)) \quad (40)$$

Recalling again Eqs. (13) and (14), the main overlap and the activity read

$$m^1(2) = \frac{1}{A} \left\langle\left\langle \xi^1 \int \mathcal{D}z z g_b(\xi^1 m^1(1) + \alpha \chi(0) \sigma(0) + \sqrt{\alpha AD(1)} z) \right\rangle\right\rangle \quad (41)$$

$$a(2) = \left\langle\left\langle \int \mathcal{D}z z g_b^2(\xi^1 m^1(0) + \alpha \chi(0) \sigma(0) + \sqrt{\alpha AD(1)} z) \right\rangle\right\rangle \quad (42)$$

and

$$\chi(1) = \frac{1}{\sqrt{\alpha AD(1)}} \left\langle\left\langle \int \mathcal{D}z z g_b(\xi^1 m^1(1) + \alpha \chi(0) \sigma(0) + \sqrt{\alpha AD(1)} z) \right\rangle\right\rangle \quad (43)$$

These equations correspond to the equations for the $Q=2$ -network found in ref. 8. The calculation of the third order parameter, i.e., the variance of the residual overlap, needs some more work. From the recursion formula (25) one finds

$$\text{Cov}[\tilde{r}^\mu(1), r^\mu(1)] = \text{Cov}[\tilde{r}^\mu(1), \tilde{r}^\mu(0)] + \chi(0) \text{Cov}[\tilde{r}^\mu(1), r^\mu(0)] \quad (44)$$

$$= \frac{1}{A} R(2, 1) + \chi(0) \frac{1}{A} R(2, 0) \quad (45)$$

with the correlation parameters, $R(t, \tilde{t})$, defined as

$$R(t, \tilde{t}) = \langle\langle \mathbf{g}_b(\hat{h}^\mu(t-1)) \mathbf{g}_b(\hat{h}^\mu(\tilde{t}-1)) \rangle\rangle \quad t, \tilde{t} \geq 1 \quad (46)$$

$$R(t, 0) = \langle\langle \sigma(0) \mathbf{g}_b(\hat{h}^\mu(t-1)) \rangle\rangle \quad t \geq 1 \quad (47)$$

This is based on the fact that by definition of the modified local field (20) $\{\xi_i^\mu, \mathbf{g}_b(\hat{h}_i^\mu(0))\}_i$ and $\{\xi_i^\mu, \sigma_i(0)\}_i$ become a set of uncorrelated variables. These results lead to the recursion relation (recall Eq. (28))

$$D(2) = \frac{a(2)}{A} + \chi^2(1) D(1) + 2 \frac{\chi(1)}{A} (R(2, 1) + \chi(0) R(2, 0)) \quad (48)$$

We still have to determine the $R(t, \tilde{t})$. The correlation $R(2, 0)$ can be written down immediately again by using the definition of the modified local field at $t = 1$

$$R(2, 0) = \left\langle\left\langle \sigma(0) \int \mathcal{D}z \mathbf{g}_b(\xi^1 m^1(1) + \alpha\chi(0)\sigma(0) + \sqrt{\alpha AD(1)} z) \right\rangle\right\rangle \quad (49)$$

To obtain $R(2, 1)$, one remarks that due to the dependence of $\sigma_i(0)$ and $\sigma_i(1)$ the local fields $h_i(1)$ and $h_i(0)$ are correlated. The correlation coefficient of their normally distributed part in general defined as

$$\rho(t, \tilde{t}) \equiv \frac{E[(h(t) - M(t))(h(\tilde{t}) - M(\tilde{t}))]}{\sqrt{V(t)} \sqrt{V(\tilde{t})}} \quad (50)$$

is found using the recursion formula (40)

$$\rho(1, 0) = \frac{\alpha R(1, 0) + \alpha A \chi(0) D(0)}{\sqrt{\alpha AD(0)} \sqrt{\alpha AD(1)}} \quad (51)$$

Employing all this in Eq. (46) we arrive at

$$R(2, 1) = \left\langle\left\langle \int \mathcal{D}w^{1,0}(x, y) \mathbf{g}_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) \right. \right. \\ \left. \left. \times \mathbf{g}_b(\xi^1 m^1(1) + \alpha\chi(0)\sigma(0) + \sqrt{\alpha AD(1)} y) \right\rangle\right\rangle \quad (52)$$

Here the joint distribution $\mathcal{D}w^{1,0}(x, y)$ equals

$$\mathcal{D}w^{1,0}(x, y) = \frac{dx dy}{2\pi \sqrt{1 - \rho(1, 0)^2}} \exp\left(-\frac{x^2 - 2\rho(1, 0)xy + y^2}{2(1 - \rho(1, 0)^2)}\right) \quad (53)$$

We remark that, for $Q=2$, the result (48) is slightly different from the corresponding result in ref. 14. Indeed, in their approach these authors make an ansatz stating the independence of the normally distributed and discrete part in the noise arising from $t \geq 2$ onwards. This results in a different expression for the correlation $R(2, 1)$. We find from Eq. (52)

$$R(2, 1) = \sum_{\sigma = \pm 1} \frac{1 + \sigma m^1(0)}{2} \int \mathcal{D}x \operatorname{sign}(m^1(0) + \sqrt{\alpha} x) \times \operatorname{Erf} \left(\frac{m^1(1) + \sigma \alpha \chi(0) + \sqrt{\alpha D(1)} \rho(1, 0) x}{\sqrt{2\alpha D(1)(1 - \rho^2(1, 0))}} \right) \quad (54)$$

with $\operatorname{Erf}(x) = (2/\sqrt{\pi}) \int_0^x \mathcal{D}y \exp(-y^2)$. The expression used by ref. 14 is obtained when we put $\rho(1, 0)$ in the above equal to zero. So, although these authors state explicitly that they have no convincing arguments in favour of (as well as against) this ansatz for $t \geq 2$, we see that their ansatz really ignores some correlations. The effects on the dynamics will be illustrated in Section 6. In our approach we do not need this ansatz.

4.3. Third Step Dynamics

We start by writing down the distribution of the local field at time $t=2$. From Eqs. (31) and (32) we find

$$h_i(2) = \xi_i^1 m^1(2) + \alpha \chi(1) [\sigma_i(1) + \chi(0) \sigma_i(0)] + \mathcal{N}(0, \alpha AD(2)) \quad (55)$$

This gives for the main overlap

$$m^1(3) = \frac{1}{A} \langle\langle \xi^1 \mathbf{g}_b(\xi^1 m^1(2) + \alpha \chi(1) [\sigma(1) + \chi(0) \sigma(0)] + \sqrt{\alpha AD(2)} y) \rangle\rangle \quad (56)$$

with y the Gaussian random variable $\mathcal{N}(0, 1)$. The average has to be taken over y , $\sigma_i(0)$ and $\sigma_i(1)$. The average over $\sigma_i(0)$ causes no difficulties because this initial configuration is chosen randomly. The average over y , the Gaussian random variable appearing in $h_i(2)$, and $\sigma_i(1)$ is more tricky because $h_i(2)$ and $\sigma_i(1)$ are correlated by the dynamics. However, the evolution equation (4) tells us that $\sigma_i(1)$ can be replaced by $\mathbf{g}_b(h_i(0))$ and, hence, the average taken over $h_i(0)$ instead of $\sigma_i(1)$.

From the recursion relation (29) one finds for the correlation coefficient between $h_i(0)$ and $h_i(2)$

$$\rho(2, 0) = \frac{\alpha [R(2, 0) + \chi(1) [R(1, 0) + \chi(0) AD(0)]]}{\sqrt{\alpha AD(0)} \sqrt{\alpha AD(2)}} \quad (57)$$

Using all this the main overlap at the third time step (56) becomes

$$m^1(3) = \frac{1}{A} \left\langle \left\langle \xi^1 \int \mathcal{D}w^{2,0}(x, y) \mathbf{g}_b(\xi^1 m^1(2) + \alpha\chi(1)[\mathbf{g}_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) + \chi(0) \sigma(0)] + \sqrt{\alpha AD(2)} y) \right\rangle \right\rangle \quad (58)$$

where the joint distribution of x and y equals

$$\mathcal{D}w^{2,0}(x, y) = \frac{dx dy}{2\pi \sqrt{1 - \rho(2, 0)^2}} \exp\left(-\frac{x^2 - 2\rho(2, 0) xy + y^2}{2(1 - \rho(2, 0)^2)}\right) \quad (59)$$

In an analogous way one arrives at the expression for the activity at the third time step

$$a(3) = \left\langle \left\langle \int \mathcal{D}w^{2,0}(x, y) \mathbf{g}_b^2(\xi^1 m^1(2) + \alpha\chi(1)[\mathbf{g}_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) + \chi(0) \sigma(0)] + \sqrt{\alpha AD(2)} y) \right\rangle \right\rangle \quad (60)$$

For $Q=2$, the result (58) again differs from what can be found in the literature, as can be seen when writing out explicitly this formula

$$m^1(3) = \sum_{\sigma=\pm 1} \frac{1 + \sigma m^1(0)}{2} \times \left\{ \int_{-m(0)/\sqrt{\alpha}}^{\infty} \mathcal{D}_x \operatorname{Erf}\left(\frac{m^1(2) + \alpha\chi(1)[1 + \sigma\chi(0)] + \sqrt{\alpha D(2)} \rho(2, 0) x}{\sqrt{2\alpha D(2)[1 - \rho^2(2, 0)]}}\right) + \int_{-\infty}^{-m(0)/\sqrt{\alpha}} \mathcal{D}_x \operatorname{Erf}\left(\frac{m^1(2) + \alpha\chi(1)[-1 + \sigma\chi(0)] + \sqrt{\alpha D(2)} \rho(2, 0) x}{\sqrt{2\alpha D(2)[1 - \rho^2(2, 0)]}}\right) \right\} \quad (61)$$

Not only is the $D(2)$ used here different as explained above but also the correlations $\rho(2, 0)$ are not taken into account in the literature.

Finally, in order to find the variance of the residual overlap at the third time step, $D(3)$, we start by rewriting Eq. (28) as

$$D(3) = \frac{a(3)}{A} + \chi^2(2) D(2) + 2 \frac{\chi(2)}{A} (R(3, 2) + \chi(1)(R(3, 1) + \chi(0) R(3, 0))) \quad (62)$$

with

$$\begin{aligned} \chi(2) = & \left\langle \left\langle \frac{1}{\sqrt{\alpha AD(2)(1 - \rho(2, 0)^2)}} \int \mathcal{D}z z \right. \right. \\ & \times \int \mathcal{D}y g_b(\xi^1 m(2) + \alpha \chi(1)(g_b(\xi^1 m(0) + \sqrt{\alpha AD(0)} y) + \chi(0) \sigma(0)) \\ & \left. \left. + \sqrt{\alpha AD(2)(1 - \rho(2, 0)^2)} z + \sqrt{\alpha AD(2)} \rho(2, 0) y \right\rangle \right\rangle \quad (63) \end{aligned}$$

Here we have used the recursion relation (25) for $r^\mu(2)$ and the fact that $\{\xi_i^\mu, g_b(\hat{h}_i^\mu(0))\}_i$ and $\{\xi_i^\mu, \sigma_i(0)\}_i$ become a collection of uncorrelated variables for $\mu \in \mathcal{P} \setminus \{1\}$. We then have to calculate the correlations $R(3, 0)$, $R(3, 1)$ and $R(3, 2)$. From the definition (46), the local field (55) and the joint distribution (59) one easily arrives at

$$\begin{aligned} R(3, 0) = & \left\langle \left\langle \sigma(0) \int \mathcal{D}w^{2,0}(x, y) g_b(\xi^1 m^1(2)) \right. \right. \\ & + \alpha \chi(1) [g_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) + \chi(0) \sigma(0)] \\ & \left. \left. + \sqrt{\alpha AD(2)} y \right\rangle \right\rangle \quad (64) \end{aligned}$$

$$\begin{aligned} R(3, 1) = & \left\langle \left\langle \int \mathcal{D}w^{2,0}(x, y) g_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) g_b(\xi^1 m^1(2)) \right. \right. \\ & + \alpha \chi(1) [g_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) + \chi(0) \sigma(0)] \\ & \left. \left. + \sqrt{\alpha AD(2)} y \right\rangle \right\rangle \quad (65) \end{aligned}$$

Finding $R(3, 2)$ is more tricky since, after rewriting the network configurations $\{\sigma_i(t)\}$ at time $t=1$ and $t=2$ by means of the gain function (4), the local fields at the three first time steps appear. So one has to calculate the elements of the correlation coefficients between these fields. The coefficients $\rho(1, 0)$ and $\rho(2, 0)$ have been obtained already before (recall Eqs. (51) and

(57)). The correlation coefficient $\rho(2, 1)$ of $h(2)$ and $h(1)$ is found by using the recursion relation (29) as

$$\rho(2, 1) = \frac{\alpha[R(2, 1) + \chi(0) R(2, 0) + \chi(1) AD(1)]}{\sqrt{\alpha AD(2)} \sqrt{\alpha AD(1)}} \quad (66)$$

The distribution function $\mathcal{D}w^{2,1}(x, y, z)$ of the three local fields equals

$$\mathcal{D}w^{2,1}(x, y, z) = \frac{dx dy dz}{(2\pi)^{3/2} \sqrt{\text{Det } w^{2,1}}} \exp\left(-\frac{1}{2}(x \ y \ z)(w^{2,1})^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) \quad (67)$$

where

$$w^{2,1} = \begin{pmatrix} 1 & \rho(1, 0) & \rho(2, 0) \\ \rho(1, 0) & 1 & \rho(2, 1) \\ \rho(2, 0) & \rho(2, 1) & 1 \end{pmatrix} \quad (68)$$

Finally, using all this information one gets for the correlation parameter

$$\begin{aligned} R(3, 2) = & \left\langle \int \mathcal{D}w^{2,1}(x, y, z) \mathbf{g}_b(\xi^1 m^1(1) + \alpha\chi(0) \sigma(0) + \sqrt{\alpha AD(1)} y) \right. \\ & \times \mathbf{g}_b(\xi^1 m^1(2) + \alpha\chi(1) [\mathbf{g}_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) + \chi(0) \sigma(0)] \\ & \left. + \sqrt{\alpha AD(2)} z) \right\rangle \quad (69) \end{aligned}$$

These results can be compared with those for extremely diluted systems. If the dilution is symmetric (see refs. 5 and 6 for the case $Q=2$) feedback loops over two time steps can exist, but the probability to have loops over a longer time period equals zero. Therefore the $\sigma_i(0)$ -term in (55) drops out. Furthermore in this symmetrically diluted $Q=2$ case the expression for the correlation coefficient (57) simply reads $\rho(2, 0) = R(2, 0)$. If the dilution is asymmetric,⁽²⁾ all feedback disappears and the local field is simply Gaussian distributed.

4.4. Fourth Step Dynamics

Again, from Eqs. (31) and (32) we find

$$\begin{aligned} h_i(3) = & \xi_i^1 m^1(3) + \alpha\chi(2) [\sigma_i(2) + \alpha\chi(1) (\sigma_i(1) \\ & + \chi(0) \sigma_i(0))] + \mathcal{N}(0, \alpha AD(3)) \quad (70) \end{aligned}$$

This leads to the main overlap for the fourth time step

$$m^1(4) = \frac{1}{A} \left\langle \left\langle \xi^1 \mathbf{g}_b(\xi^1 m^1(3) + \alpha \chi(2) [\sigma(2) + \alpha \chi(1) (\sigma(1) + \chi(0) \sigma(0))] + \sqrt{\alpha AD(3)} z) \right\rangle \right\rangle \quad (71)$$

with z the Gaussian random variables $\mathcal{N}(0, 1)$. The average has to be taken over z , $\sigma_i(0)$, $\sigma_i(1)$, and $\sigma_i(2)$ or recalling the evolution equation (4) over z , $\sigma_i(0)$, $h_i(0)$ and $h_i(1)$. The distribution function of these variables, i.e., $\mathcal{D}w^{3,1}(x, y, z)$ is given by Eqs. (67)–(68) with the index 2 replaced by 3. The correlation coefficients between the fields at different time steps can again be calculated from the recursion relation (29). The following coefficients not written down before are needed

$$\rho(3, 1) = \frac{\left(\alpha [\chi(2)(\chi(1) D(1) + \chi(0) R(2, 0) + R(2, 1)) + \chi(0) R(3, 0) + R(3, 1)] \right)}{\sqrt{\alpha AD(1)} \sqrt{\alpha AD(3)}} \quad (72)$$

$$\rho(3, 0) = \frac{\alpha [\chi(2)(R(2, 0) + \chi(1)(R(1, 0) + \chi(0))) + R(3, 0)]}{\sqrt{\alpha AD(0)} \sqrt{\alpha AD(3)}} \quad (73)$$

$$R(3, 1) = \left\langle \left\langle \int \mathcal{D}w^{2,0}(x, y) \mathbf{g}_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) \times \mathbf{g}_b(\xi^1 m^1(2) + \alpha \chi(1) [\mathbf{g}_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) + \chi(0) \sigma(0)]) + \sqrt{\alpha AD(2)} y \right\rangle \right\rangle \quad (74)$$

Using all this Eq. (71) becomes

$$m^1(4) = \frac{1}{A} \left\langle \left\langle \xi^1 \int \mathcal{D}w^{3,1}(x, y, z) \mathbf{g}_b(\xi^1 m^1(3) + \alpha \chi(2) \{ \mathbf{g}_b(\xi^1 m^1(1) + \alpha \chi(0) \sigma(0) + \sqrt{\alpha AD(1)} y) + \chi(1) [\mathbf{g}_b(\xi^1 m^1(0) + \sqrt{\alpha AD(0)} x) + \chi(0) \sigma(0)] \} + \sqrt{\alpha AD(3)} z) \right\rangle \right\rangle \quad (75)$$

In an analogous way the activity at the fourth time step can be calculated.

Since the numerical simulations explicitly show that the third time step gives already an accurate picture of the dynamics in the retrieval region we only write out the relevant formula here for $Q=2$. The idea thereby is to find out how the effect of the ansatz neglecting some correlations as discussed at several instances above propagates in time. We find

$$\begin{aligned}
m^1(4) = & \sum_{\sigma=\pm 1} \frac{1+\sigma m^1(0)}{2} \int \mathcal{D}w^{3,1}(x, y, z) \text{sign}(m^1(3)) \\
& + \alpha\chi(2)[\text{sign}(m^1(1) + \sigma\alpha\chi(0) + \sqrt{\alpha D(1)} y) \\
& + \chi(1)[\text{sign}(m^1(0) + \sqrt{\alpha D(0)} x) + \sigma\chi(0)]] + \sqrt{\alpha D(3)} z) \quad (76)
\end{aligned}$$

This formula for $Q=2$ has not been given explicitly in the literature even not in the approximation proposed in ref. 14. The result of this approximation is to take the matrix $w^{3,1}$ in the formula (76) above to be the unit matrix and to use the approximate form for $D(3)$ mentioned before. The effect of this will be discussed further in Section 6.

5. FIXED-POINT EQUATIONS

A second type of results can be obtained by requiring through the recursion relations (28), (31) and (32) that the local field becomes time-independent. This means that most of the discrete noise part is neglected. We show that this procedure leads to the same fixed-point equations as those found from thermodynamics in ref. 21.

For the Q -Ising model at zero temperature one can show that

$$H(t) = -\frac{1}{4} \sum_{i \in \mathcal{A}} \left(\sum_{j \in \mathcal{A}} J_{ij} \sigma_j(t) \tilde{\sigma}_i(t) - b(\tilde{\sigma}_i^2(t) + \sigma_i^2(t)) \right) \quad (77)$$

with $\tilde{\sigma}_i(t)$ chosen such that

$$\varepsilon_i[\tilde{\sigma}_i(t) | \sigma_{\mathcal{A} \setminus \{i\}}(t)] = \min_{s \in \mathcal{S}} \varepsilon_i[s | \sigma_{\mathcal{A} \setminus \{i\}}(t)] \quad (78)$$

is a Lyapunov function. For finite N , $H(t)$ is bounded from below implying that $H(t+1) - H(t) = 0$ after finitely many time steps. This can be realized for $\sigma_i(t+2) = \sigma_i(t) \forall i \in \mathcal{A}$. The proof is straightforward and completely analogous to the argumentation used in refs. 17 and 18. Both a fixed point and a two-cycle satisfy this condition. As stated in the introduction we only study fixed-points.

Since the evolution equations for the order parameters in the extremely diluted and layered Q -Ising models do not change their form as

time progresses, the fixed-point equations are obtained immediately by leaving out the time dependence (see refs. 2 and 4). This still allows small fluctuations in the configurations $\{\sigma_i\}$.

Since in the fully connected model treated here the form of the evolution equations for the order parameters do change by the explicit appearance of the $\{\sigma_i(t)\}$, $t \geq 0$, we can not use that procedure to obtain the fixed-point equations. Instead we require that the distribution of the local field given by (29) becomes independent of time. This is a stronger condition because fluctuations in the network configuration are no longer allowed. In fact, it means that out of the discrete part of this distribution, i.e., $M_i(t)$ (recall (33)), only the $\sigma_i(t-1)$ term is kept besides, of course, the signal term. This procedure implies that the main overlap and activity in the fixed-point are found from the definitions (7), (8) and not from leaving out the time dependence in the recursion relation (13) and (14). The same line of reasoning is followed in, e.g., refs. 24 and 25.

We start by eliminating the time-dependence in the evolution equations for the local field (29). This leads to

$$h_i = \zeta_i^1 m^1 + \frac{1}{1-\chi} \mathcal{N}(0, \alpha\alpha) + \alpha \frac{\chi}{1-\chi} \sigma_i \quad (79)$$

with $h_i \equiv \lim_{t \rightarrow \infty} h_i(t)$. This expression consists out of two parts: A normally distributed part $\tilde{h}_i = \mathcal{N}(\zeta_i^1 m^1, \alpha\alpha/(1-\chi)^2)$ and some discrete noise part. We remark that this discrete noise coming from the correlation of the $\{\sigma_i(t)\}$ at different time steps (here only the preceding time step is considered) is inherent in the fully connected dynamics.

Employing the expression Eq. (79) in the updating rule (4) one finds

$$\sigma_i = g_b(\tilde{h}_i + \alpha\eta\sigma_i) \quad \eta = \chi/(1-\chi) \quad (80)$$

This is a self-consistent equation in σ_i which in general admits more than one solution. This type of equation has been solved in the case of analog neural networks with continuous time dynamics using a Maxwell construction. Such a construction is standard in thermodynamics in order to maximize the exponent of the integrand appearing in free energy calculations. Here we use a similar geometrical construction to treat Eq. (80).

Let L be the straight line which connects the centers of the plateaus of the gain function $g_b(\cdot)$. The equations for the functions $g_b(\cdot)$ and $L(\cdot)$ read

$$g_b: x \mapsto s_k \quad \text{if } b(s_k + s_{k-1}) < x < b(s_k + s_{k+1}) \quad (81)$$

$$L: x \mapsto \frac{x}{2b} \quad (82)$$

The condition on the r.h.s. of (81) is a condition on x . Using the definition of $L(\cdot)$, one can transform this into a condition on the image of $L(\cdot)$, $\mathcal{I}_L = \{y \in \mathbb{R} \mid \exists x \in \mathbb{R} : L(x) = y\}$, viz.

$$\mathbf{g}_b(x) = s_k \quad \text{if} \quad \frac{s_k + s_{k-1}}{2} < L(x) < \frac{s_k + s_{k+1}}{2} \quad (83)$$

Consider the transformation $\mathcal{T} : (x, y) \mapsto (x - \alpha\eta y, y)$

$$\mathcal{T}(L) : x \mapsto \frac{x}{2(b - \alpha\eta/2)} \quad (84)$$

The function $\mathcal{T}(\mathbf{g}_b)(\cdot)$ is not bijective while \mathcal{T} is not one-to-one. To obtain a unique solution for Eq. (80) we modify the former function such that it becomes a step function with the same step height as the one in $\mathcal{T}(\mathbf{g}_b)(\cdot)$ and the width of the steps such that $\mathcal{T}(L)$ connects the centers of the plateaus:

$$\mathcal{T}_L(\mathbf{g}_b)(x) = s_k \quad \text{if} \quad \left(b - \frac{\alpha\eta}{2}\right)(s_k + s_{k-1}) < x < \left(b - \frac{\alpha\eta}{2}\right)(s_k + s_{k+1}) \quad (85)$$

or, using (4)

$$\mathcal{T}_L(\mathbf{g}_b)(x) = \mathbf{g}_{\tilde{b}}(x) \quad \text{with} \quad \tilde{b} = b - \frac{\alpha\eta}{2} \quad (86)$$

This at first sight ad-hoc modification leads us to a unique solution of the self-consistent equation (80). Indeed, from this modified transformation we know that

$$\mathbf{g}_b(\tilde{h} + \alpha\eta\sigma) \simeq \mathbf{g}_{\tilde{b}}(\tilde{h} + \alpha\eta\sigma - \alpha\eta\mathbf{g}_b(\tilde{h} + \alpha\eta\sigma)) \quad (87)$$

such that

$$\sigma_i = \mathbf{g}_{\tilde{b}}(\tilde{h}_i) \quad (88)$$

At this point we remark that plugging this result into the local field equation (79) tells us that the latter is the sum of two Gaussians with shifted mean (see also ref. 12).

Using the definition of the main overlap and activity (7) and (8) in the limit $N \rightarrow \infty$, one finds in the fixed point

$$m^1 = \left\langle\left\langle \xi^1 \int \mathcal{D}z g_{\tilde{b}}(\xi^1 m^1 + \sqrt{\alpha AD} z) \right\rangle\right\rangle \quad (89)$$

$$a = \left\langle\left\langle \int \mathcal{D}z g_{\tilde{b}}^2(\xi^1 m^1 + \sqrt{\alpha AD} z) \right\rangle\right\rangle \quad (90)$$

From (25), (28) and (27) it is clear that

$$D = \frac{a/A}{(1-\chi)^2} \quad (91)$$

with

$$\chi = \frac{1}{\sqrt{\alpha AD}} \left\langle\left\langle \int \mathcal{D}z z g_{\tilde{b}}(\xi^1 m^1 + \sqrt{\alpha AD} z) \right\rangle\right\rangle \quad (92)$$

These resulting equations (89)–(91) are the same as the fixed-point equations derived from a replica-symmetric mean-field theory treatment in ref. 21. Their solution leads to the $\alpha - b$ phase diagram Fig. 1b in ref. 21. We end with the observation that for analog networks the construction (81)–(86) is not necessary: the fixed-point equation (80) has only one solution.

6. NUMERICAL SIMULATIONS

As an illustrative example the equations derived in Section 4 have been worked out explicitly for a uniform distribution of patterns in the case of the $Q=2$ model ($A=1$) and the $Q=3$ model with equidistant states ($A=2/3$).

For the $Q=2$ model the temperature-capacity phase diagram following from a thermodynamic replica-symmetric mean-field theory approach is well known.⁽²⁶⁾ Also, there have been a lot of numerical simulations of the dynamics using approximate theories mentioned in the introduction.⁽¹⁵⁾ Therefore we mainly restrict ourselves to a comparison of the complete recursive scheme developed here and the results obtained in ref. 14 (extended to the fourth time step) based upon the ansatz discussed in Sections 4.2–4.4.

At zero temperature the critical capacity is 0.14. We present in Fig. 1 the overlap $m^1(t)$, $t=1$ to 4 versus the initial overlap m_0^1 with the condensed pattern for a typical case $\alpha=0.1$. (We forget about the superscript 1). For $m_0 \geq 0.4$ we see that the $m=1$ retrieval attractor is reached quickly. In fact three time steps give us already an accurate picture of the dynamics in the retrieval region. The results obtained by the evolution equations derived

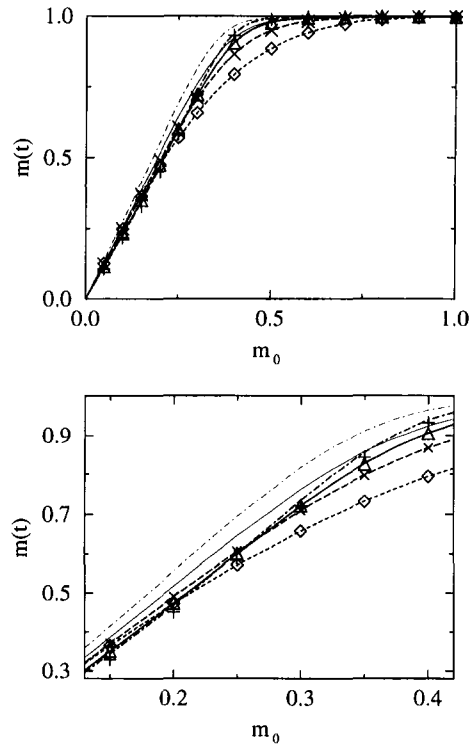


Fig. 1. A comparison of the theoretical results and numerical simulations for $Q=2$ systems with $N=6000$. The overlap $m(t)$ is presented for the first four time steps as a function of m_0 for $\alpha=0.1$. Theoretical (simulations) results for the first, second, third and fourth time step are indicated by a short-dashed curve (diamond symbol), a long-dashed curve (times symbol), a full line (triangle symbol) and a dashed-dotted curve (plus symbol) respectively. The results using ref. 14 are indicated with a thin full and a thin dashed dotted line.

here agree very well with simulations done for system-sizes up to $N=6000$ (each data point is averaged over 1000 runs). The results obtained using ref. 14 systematically overestimate the overlap for $m_0 \geq 0.24$ and this effect becomes stronger when going from time step 3 to 4. For smaller m_0 the overestimate is even in the wrong direction of the dynamical evolution.

For the $Q=3$ model a thermodynamic replica-symmetric mean-field theory approach leads to a capacity-gain phase diagram discussed already in ref. 21 (Fig. 1b). As explained in Section 5 the same phase diagram can be obtained through the dynamical approach presented here. For convenience and completeness this phase diagram is reproduced here as Fig. 2. At this point it is also useful to recall that there are two types of retrieval

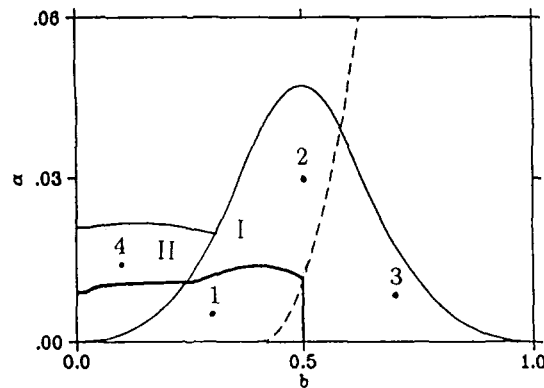


Fig. 2. The α - b phase diagram.⁽²¹⁾ The (thin) full curve indicates the maximal storage capacity, the thick full curve the thermodynamic transition of the retrieval state, the broken curve the spin-glass transition. In region I, $r \approx O(1)$ while in region II $r \approx O(10)$. The points 1 to 4 indicate the network parameters used in the numerical simulations.

states. In region I the mean-square random overlap with the non-condensed patterns, r , is of order $O(1)$ while in region II r is of order $O(10)$.⁽²¹⁾

For specific network parameters corresponding to different points in the retrieval region of this equilibrium phase diagram, indicated as 1 to 4, we have compared the dynamics governed by the evolution equations found here with extensive simulations involving system-sizes up to $N = 6000$ (each data point is averaged over 1600 runs).

Figures 3-6 present an overview of these results by plotting the overlap $m(t)$, the activity $a(t)$ and the Hamming distance $d(t)$ versus the initial overlap m_0 with the condensed pattern. The initial activity is taken to be $a_0 = 0.85$.

First we consider region I. For network parameters corresponding to point 1 below the thermodynamic transition line, i.e., $\alpha = 0.005$, $b = 0.3$, we see in Fig. 3 that for $m_0 \geq 0.33$ the dynamics quickly evolves to an overlap $m = 1$ and that the Hamming distance is zero for $m_0 \geq 0.37$. The activity attains the value $2/3$, meaning that the network configuration is uniformly distributed. The boundary between the $m = 1$ attractor and the zero-attractor is rather sharply determined.

For a network corresponding to point 2 above the thermodynamic transition line, with $\alpha = 0.03$, $b = 0.5$, we need a larger value of m_0 to reach the $m = 1$ attractor and a Hamming distance zero. As seen in Fig. 4, m_0 has to be at least 0.75. Also the boundary between the $m = 1$ attractor and the zero-attractor is less sharply determined. Figure 5 shows that this behavior is qualitatively the same for $\alpha = 0.009$, $b = 0.7$, corresponding to point 3

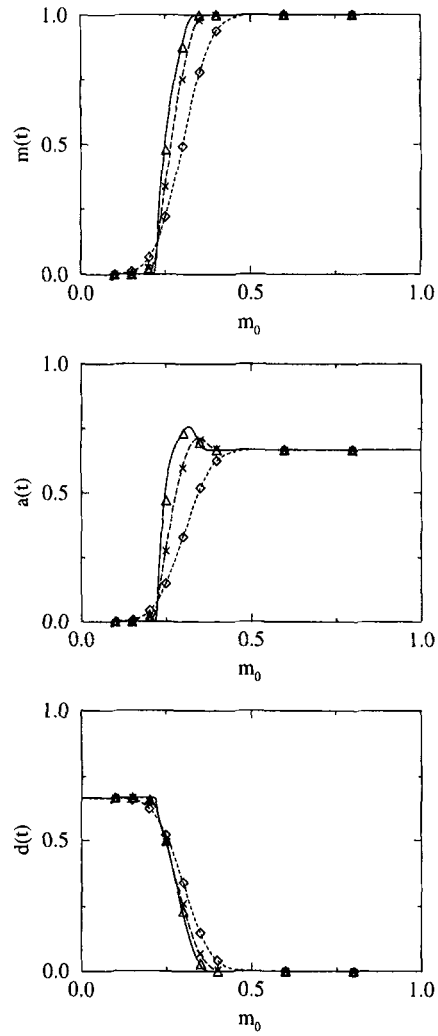


Fig. 3. A comparison of the theoretical results and numerical simulations for systems with $N = 6000$. The overlap $m(t)$, the activity $a(t)$ and the Hamming distance $d(t)$ are presented for the first three time steps as a function of m_0 for the network parameters $b = 0.3$, $\alpha = 0.005$, $a_0 = 0.85$. Theoretical (simulations) results for the first, second and third time step are indicated by a short-dashed curve (diamond symbol), a long-dashed curve (times symbol) and a full line (triangle symbol) respectively.

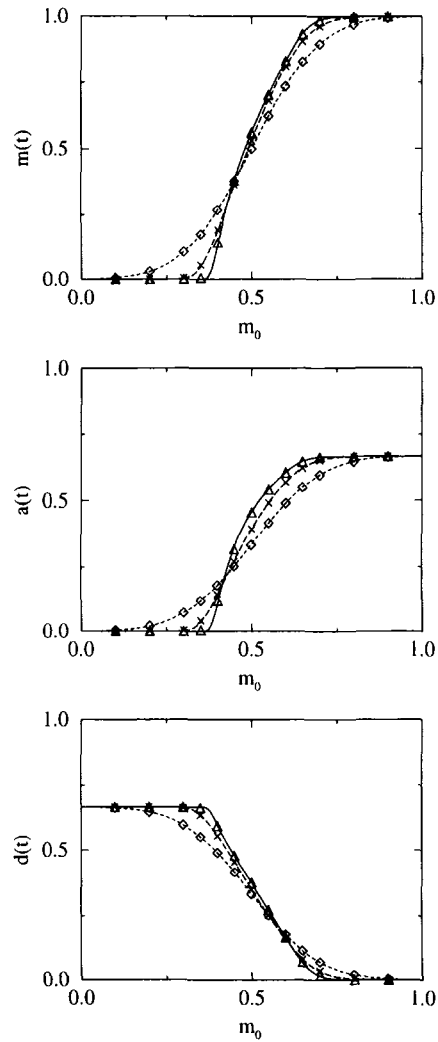


Fig. 4. As in Fig. 3, for the network parameters $b = 0.5$, $\alpha = 0.03$, $a_0 = 0.85$.

situated above the spin-glass transition in the phase diagram. In this case the value of m_0 has to be at least 0.85. For the other network parameters we have looked at, e.g., $\alpha = 0.0115$, $b = 0.5$; the global behavior is similar.

For network parameters corresponding to points in region II of the phase diagram, e.g., point 4 with $\alpha = 0.015$, $b = 0.1$ it is shown in Fig. 6 that the main overlap goes to its maximum value for almost all values of m_0 .

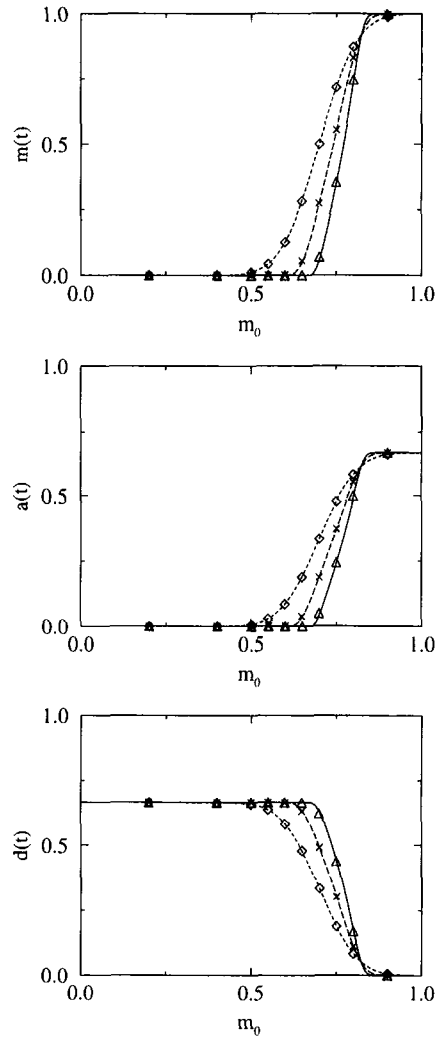


Fig. 5. As in Fig. 3, for the network parameters $b = 0.7$, $\alpha = 0.009$, $a_0 = 0.85$.

The basin of attraction of the zero fixed-point is zero. The activity, however, goes to a value larger than $2/3$. The network configuration is no longer uniformly distributed: the state $\sigma_i = 0$ has a smaller probability to appear than the states $\sigma_i = \pm 1$. Hence, the Hamming distance is never zero. This must be due to the fact that the influence of the non-condensed patterns is much larger here ($r \approx O(10)$). The same qualitative behavior is

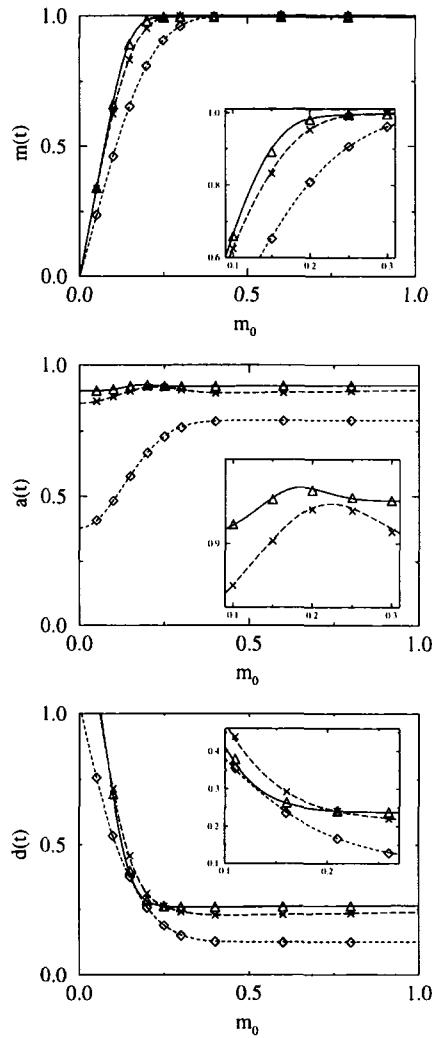


Fig. 6. As in Fig. 3, for the network parameters $b=0.1$, $\alpha=0.015$, $u_0=0.85$.

found for network parameters corresponding to points in region II below the thermodynamic transition line, e.g., $\alpha=0.005$, $b=0.1$.

7. CONCLUDING REMARKS

In this paper we have derived the evolution equation for the distribution of the local field governing the parallel dynamics at zero temperature

of fully connected $Q \geq 2$ -Ising networks. Hereby we have taken into account, for the first time, *all* feedback correlations without any approximation. In contrast with extremely diluted asymmetric and layered networks we show that this distribution is no longer normally distributed but contains a discrete part. Specializing to the $Q = 2$ case treated in the literature before by making an ansatz stating the independence of the normally distributed and discrete part of the noise arising from $t \geq 2$ onwards, we find that this ansatz ignores some correlations.

Using this evolution equation we have developed a general recursive scheme which allows us to calculate the relevant order parameters of the system, i.e., the main overlap, the activity and the variance of the residual overlap, for *any* time step. We have worked out this scheme explicitly for the first four time steps of the dynamics.

Under the condition that the local field becomes time-independent, meaning also that most of the discrete noise part is neglected, we have obtained the fixed-point equations for these order parameters. They are found to be the same as those derived via thermodynamic methods.⁽²¹⁾

As an illustration we have presented a detailed discussion of these results for the $Q = 2$ and $Q = 3$ model and we have made a comparison with extensive numerical simulations. It is seen that these numerical results provide excellent support for our theoretical predictions and that the first three time steps do give already an accurate picture of the time evolution in the retrieval regime of the network. For $Q = 2$ we also find that the ansatz mentioned above overestimates the retrieval overlap.

ACKNOWLEDGMENTS

This work has been supported in part by the Research Fund of the K.U. Leuven (Grant OT/94/9) and the Korea Science and Engineering Foundation through the SRC program. The authors are indebted to S. Amari, R. Kühn, A. Patrick, and V. Zagebnov for constructive discussions. One of us (D.B.) thanks the Belgian National Fund for Scientific Research for financial support.

REFERENCES

1. B. Derrida, E. Gardner, and A. Zippelius, *Europhys. Lett.* **4**:167 (1987).
2. D. Bollé, G. M. Shim, B. Vinck, and V. A. Zagebnov, *J. Stat. Phys.* **74**:565 (1994).
3. E. Domany, W. Kinzel, and R. Meir, *J. Phys. A: Math. Gen.* **22**:2081 (1989).
4. D. Bollé, G. M. Shim, and B. Vinck, *J. Stat. Phys.* **74**:583 (1994).
5. T. L. H. Watkin and D. Sherrington, *J. Phys. A: Math. Gen.* **24**:5427 (1991).
6. A. E. Patrick and V. A. Zagebnov, *J. Phys. A: Math. Gen.* **23**:L1323 (1990); **25**:1009 (1992).

7. W. Kinzel, *Z. Phys. B* **60**:205 (1985).
8. E. Gardner, B. Derrida, and P. Mottishaw, *J. Physique* **48**:741 (1987).
9. S. Amari and K. Maginu, *Neural Networks* **1**:63 (1988).
10. W. Krauth, J. P. Nadal, and M. Mezard, *J. Phys. A: Math. Gen.* **21**:2995 (1988).
11. H. Horner, D. Bormann, M. Frick, H. Kinzelbach, and A. Schmidt, *Z. Phys. B* **76**:381 (1989).
12. R. D. Henkel and M. Opper, *Europhys. Lett.* **11**:403 (1990); *J. Phys. A: Math. Gen.* **24**:2201 (1991).
13. A. E. Patrick and V. A. Zagrebnoy, *J. Stat. Phys.* **63**:59 (1991).
14. A. E. Patrick and V. A. Zagrebnoy, *J. Phys. A: Math. Gen.* **24**:3413 (1991).
15. M. Okada, *Neural Networks* **9**:1429 (1996).
16. D. Bollé, B. Vinck, and V. A. Zagrebnoy, *J. Stat. Phys.* **70**:1099 (1993).
17. P. Peretto, *Biol. Cybern.* **50**:51 (1984).
18. J. L. van Hemmen and R. Kühn, in *Models of Neural Networks*, E. Domany, J. L. van Hemmen, and K. Schulten, eds. (Springer, 1991), p. 1.
19. J. F. Fontanari and Köberle, *J. Physique* **49**:13 (1988).
20. T. Stiefvater, K. R. Müller, and R. Kühn, *Physica A* **232**:61 (1996).
21. D. Bollé, H. Rieger, and G. M. Shim, *J. Phys. A: Math. Gen.* **27**:3411 (1994).
22. E. Barkai, I. Kanter, and H. Sompolinsky, *Phys. Rev. A* **41**:590 (1990).
23. A. N. Shiryaev, *Probability* (Springer, New York, 1984).
24. M. Shiino and T. Fukai, *J. Phys. A: Math. Gen.* **25**:L375 (1992).
25. M. Shiino and T. Fukai, *Phys. Rev. E* **48**:867 (1993).
26. D. Amit, H. Gutfreund, and H. Sompolinsky, *Ann. Phys. (N.Y.)* **173**:30 (1987).